Tensor Product Representations on Riemannian Structures.

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1 Abstract:

THE research paper covers and explores several mathematical concepts of Riemannian Manifolds and Structures in Abstract Algebra and Topology. We prove that:

$$\int_{\mathcal{P}_{S,F}} \mathcal{F}_t(i,j,k)\, dV = \int_S^F \left(\int_{\mathcal{P}_t} \mathcal{T}_i(t) \otimes \mathcal{T}_{ij}(t) \otimes \mathcal{T}_{ijk}(t)\, dV_t \right) dt = \mathcal{T}_i \otimes \mathcal{T}_{ij} \otimes \mathcal{T}_{ijk} = \langle \mathcal{P} \rangle. \tag{1}$$

And consequently additional remarks and observations regarding tensor products and their representations over Riemannian Manifolds are discussed using the Fubini's Theorem and a proof by construction. Previous knowledge on Abstract Algebra, and Topology is assumed.

Keywords: Topology, Abstract Algebra, Machine Learning.

2 Introduction and Definitions:

THE following definition of $\mathcal{P}$ is a p-dimensional Riemannian manifold product tensor with boundary. One may define such as:

2.1 Definitions of a Tensor and a Tensor Set:

Let a tensor $\mathcal{X}$ be defined as in:

$$\mathcal{X} \subset \mathcal{T}_i \otimes \mathcal{T}_{ij} \otimes \mathcal{T}_{ijk}. \tag{2}$$

For the following tensor $\mathcal{T}_i \otimes \mathcal{T}_{ij} \otimes \mathcal{T}_{ijk}$. to be defined as in:

$$\mathcal{T}_i \otimes \mathcal{T}_{ij} \otimes \mathcal{T}_{ijk}. \in \mathbb{T}^+. \tag{3}$$

Where $\mathbb{T}^+$ is the set of all non-zero three-dimensional tensors such as $\mathcal{T}_i \otimes \mathcal{T}_{ij} \otimes \mathcal{T}_{ijk}$.

2.2 Definitions of a Riemannian Manifold Product Tensor:

Definition 2.1. Let $\mathcal{P}$ denote a p-dimensional Riemannian manifold product with boundary. We will always assume $\mathcal{P}$ to factor three three-dimensional tensors that are non-zero. One may define such as:

$$\langle \mathcal{P} \rangle = \mathcal{X}. \tag{4}$$

A general identity, defined as an integral over the subregion $\mathcal{P}_{S,F}$ with respect to the Riemannian volume form, where $t \in [S,F]$ is a foliation parameter indexing a family of hypersurfaces $\{\mathcal{P}_t\}_{t\in[S,F]}$ that foliates $\mathcal{P}_{S,F}$, and i,j,k are continuous local coordinates on each leaf $\mathcal{P}_t$:

$$\langle \mathcal{P} \rangle_g = \int_{\mathcal{P}_{S,F}} \mathcal{F}_t(i,j,k)\, dV. \tag{5}$$

Where V is the Riemannian volume form on $\mathcal{P}$. And g represents a discrete part of the variable $\mathcal{P}$.

3 Additional Remarks regarding our Proofs:

IN this section, several remarks will be stated here regarding our observations of our theorems that have been written per Definition 2.1.

Remark 3.1. There exist degenerate cases $\mathcal{P}_d$ of $\mathcal{P}$ for which the integral method does not converge in $\mathbb{R}$.

4 Proofs of Definition 2.1 over Riemannian Manifolds:

4.1 Definitions of Corollaries for Manifolds:

Corollary 4.1. *Let $\mathcal{F}_t$ be a continuous tensor field on the compact region $\mathcal{P}_{S,F} \subset \mathcal{P}$. Then $\mathcal{F}_t$ is integrable with respect to the Riemannian volume form dV, and $\int_{\mathcal{P}_{S,F}} \mathcal{F}_t(i,j,k)\,dV$ is well-defined.*

Theorem 4.2. *Per Definition 2.1, the integral $\int_{\mathcal{P}_{S,F}} \mathcal{F}_t(i,j,k)\,dV$ defines a specific case of $\langle\mathcal{P}\rangle$ over a Riemannian manifold, where $\mathcal{F}_t$ is a continuous tensor field on $\mathcal{P}$ and $\mathcal{P}_{S,F}$ is the subregion bounded by regions S and F. This follows from Corollary 4.1 and Fubini's theorem for Riemannian submersions: if $\mathcal{P}_{S,F}$ admits a foliation by t-slices, the integral decomposes as an iterated integral over fibers, recovering the contraction $\mathcal{T}_i \otimes \mathcal{T}_{ij} \otimes \mathcal{T}_{ijk}$.*

Corollary 4.3. *Per definitions in Definition 2.1, since the tensors are assumed non-zero ($\langle\mathcal{P}\rangle$), there are non degenerate cases of Definition 2.1 for both of the equations as restated below:*

$$\langle\mathcal{P}\rangle = \mathcal{T}_i \otimes \mathcal{T}_{ij} \otimes \mathcal{T}_{ijk}. \tag{6}$$

And thus:

$$\langle\mathcal{P}\rangle = \int_{\mathcal{P}_{S,F}} \mathcal{F}_t(i,j,k)\,dV. \tag{7}$$

Corollary 4.4. *Per $\mathcal{F}_t(i,j,k)$ being factorizable as $\mathcal{T}_i(t) \otimes \mathcal{T}_{ij}(t) \otimes \mathcal{T}_{ijk}(t)$ on each leaf $\mathcal{P}_t$ of the foliation. There exist a conjugate variant of $\mathcal{P}_t$.*

4.2 Proof by construction regarding Corollaries and Definitions:

Proof. We proceed by construction, invoking Corollaries 4.3 and 4.4.
(i) By Corollary 4.3, the contraction $\mathcal{T}_i \otimes \mathcal{T}_{ij} \otimes \mathcal{T}_{ijk}$ is non-degenerate on $\mathcal{P}_{S,F}$, since the constituent tensors are assumed non-zero by Definition 2.1. In particular, $\langle\mathcal{P}\rangle \neq 0$ on the initial leaf $\mathcal{P}_S$.
(ii) By Corollary 4.4, $\mathcal{F}_t(i,j,k)$ factors as $\mathcal{T}_i(t) \otimes \mathcal{T}_{ij}(t) \otimes \mathcal{T}_{ijk}(t)$ on each leaf $\mathcal{P}_t$. Suppose the identity $\langle\mathcal{P}\rangle = \int_{\mathcal{P}_{S,t}} \mathcal{F}_t(i,j,k)\,dV$ holds for all leaves up to $\mathcal{P}_t$. Since $\mathcal{F}_t$ is continuous and non-degenerate by Corollaries 4.3 and 4.4, the identity extends to $\mathcal{P}_{t+\epsilon}$ for any $\epsilon > 0$ by continuity of the foliation parameter.
(iii) Applying Fubini's theorem for Riemannian submersions over $\mathcal{P}_{S,F}$:

$$\int_{\mathcal{P}_{S,F}} \mathcal{F}_t(i,j,k)\,dV = \int_S^F \left(\int_{\mathcal{P}_t} \mathcal{T}_i(t) \otimes \mathcal{T}_{ij}(t) \otimes \mathcal{T}_{ijk}(t)\,dV_t\right) dt = \mathcal{T}_i \otimes \mathcal{T}_{ij} \otimes \mathcal{T}_{ijk} = \langle\mathcal{P}\rangle. \tag{8}$$

(iv) Since the factorization is non-degenerate on every leaf and Fubini assembles the leaves globally over $\mathcal{P}_{S,F}$, both equations of Definition 2.1 are simultaneously satisfied, completing the proof. □

5 Conclusion:

One may expand the observations and experimentation in Analysis and Complex Analysis. Moreover, per our abstract: We covered several products and their proofs over a Riemannian Manifold.

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